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Journal of Functional Analysis 205 (2003) 90–106

JOURNAL OF Functional Analysis

http://www.elsevier.com/locate/jfa

A characterization of the closed unital ideals of the Fourier–Stieltjes algebra $B(G)$ of a locally compact amenable group $G^{\star}$

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Received 7 August 2002; accepted 3 January 2003

Communicated by D. Sarason

Cet article est dedié au Professeur P. Eymard en reconnaissance de ce qu'il a fait pour moi quand j'étais étudiant à L'Université de Nancy

Abstract

Let G be a locally compact amenable group, $B(G)$ its Fourier–Stieltjes algebra and I be a closed ideal of it. In this paper we prove the following result: The ideal I has a unit element iff it is principal. This is the noncommutative version of the Glicksberg–Host–Parreau Theorem. The paper also contains an abstract version of this theorem.

MSC: primary 47B48; 46J10; 43A22

Keywords: Multiplier; Fourier algebra; Fourier–Stieltjes algebra

1. Introduction

Let G be a locally compact abelian group, $L^1(G)$ be its group algebra and $M(G)$ be its measure algebra. Let $\mu \in M(G)$ be a given measure. If μ is the product of an invertible measure λ and an idempotent measure θ (i.e. $\mu = \lambda * \theta$) then the ideal $\mu * L^1(G)$ is obviously closed in $L^1(G)$. The problem whether the converse is also true, which was raised by Hewitt, has been first studied by Glicksberg [5] and later by

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doi:10.1016/S0022-1236(03)00143-5

Host and Parreau, who completely solved it in their quite impressive paper [7]. Thus $\mu * L^1(G)$ is closed in $L^1(G)$ iff μ is the product of an invertible measure λ and an idempotent measure θ. This is the Glicksberg–Host–Parreau Theorem. It is easy to see (Lemma 2.1) that this theorem is equivalent to the statement: A closed ideal I of $M(G)$ is unital iff it is of the form $I = \mu * M(G)$ for some measure $\mu \in M(G)$. As far as we know the literature on the subject, the noncommutative version of this theorem is not known. Our aim in this paper is to establish the analogue of the Glicksberg–Host–Parreau Theorem for the Fourier algebra $A(G)$ of a locally compact amenable group G. The proof given by Host and Parreau, which is the only proof available of this theorem as of the day, is highly abelian-group theoretical and it does not seem possible to adapt it to noncommutative case. So we have had to take a completely different path. Our proof is entirely functional analytic, elementary and self-contained. We first establish the following general result, which is one of our main results: Let X be a Banach space and $L : X \to X$ be a continuous linear operator with closed range. Suppose that $B = X^*$ is a commutative semisimple unital Banach algebra under some multiplication and that L^* is a multiplier on it. Then the ideal $L^*(B)$ has a unit element iff $Ker(L^2) = Ker(L)$. To see the relevance of this result to the result stated in the abstract, let G be a locally compact amenable group and $X = C^*(G)$ be the group C^*-algebra of G. Then $X^* = B(G)$, the Fourier–Stieltjes algebra of the group G. This is a commutative unital semisimple Banach algebra. Moreover, for $u \in B(G)$ fixed, since the multiplication operator $v \mapsto uv$ is continuous on $B(G)$ for the weak-star topology of this space, $B(G)u = L^*(B(G))$ for some continuous linear operator $L : C^*(G) \to C^*(G)$ such that $L^*(v) = vu$. Now suppose that the ideal $I = B(G)u$ is closed in $B(G)$. The preceding result says that the ideal $B(G)u$ is unital iff we have $Ker(L^2) = Ker(L)$. As for the equality $Ker(L^2) = Ker(L)$, it is a consequence of the fact that the Fourier algebra $A(G)$ of G is Tauberian (Lemma 2.6). If we take a commutative locally compact group G and $X = C_0(G)$, then we obtain the Glicksberg–Host–Parreau Theorem in its full generality. In Section 3, we present an abstract version of the Glicksberg–Host–Parreau Theorem. To this end, let A be a commutative semisimple, regular Tauberian Banach algebra with a BAI (= bounded approximate identity) and that every open subset of $\Delta(A)$ is a u-set with respect to the algebra A. (This notion is defined in the text and also in [16].) Let $A^*A = \{fa : a \in A \ and \ f \in A^*\}$, where fa is the functional defined on A by $\langle fa, b\rangle = \langle f, ab\rangle$. This is a closed subspace of A^* [9, 32.22]. Suppose that the multiplier algebra $M(A)$ of A can be identified with the dual of a closed subspace X of A^*A, which is invariant under each T^* ($T \in M(A)$). Then, for $T \in M(A)$, the ideal $T(A)$ is closed in A iff T factors as the product of an invertible multiplier and an idempotent multiplier. Equivalently, the closed unital ideals of $M(A)$ are exactly the principal ones (i.e. The closed ideals of the form $T \circ M(A) = \{T \circ S : S \in M(A)\}$ for some $T \in M(A)$). This result (Theorem 3.7), which is the second main result of the paper, can be considered as an abstract form of the Glicksberg–Host–Parreau Theorem. The following simple example shows that the hypotheses imposed on A in the main (abstract) results are necessary. Indeed, let A be the disk algebra; the algebra A is commutative semisimple and unital. As A is unital, $M(A) = A$. Let $u(z) = z$. Then $Au = \{a \in A : a(0) = 0\}$. The ideal $Au = M(A)u$ is closed in $M(A)$ but is not unital. The reader will observe here that not only the ideal Au but the ideal $Au^2 = \{a \in A : a'(0) = a(0) = 0\}$ is also closed in A and that, in spite of this fact, $\delta(u) = \inf\{|u(z)| : |z| \leqslant 1 \textit{ and } u(z) \neq 0\} = 0$. The algebra $M(A) = A$ is of course not a dual space; nor is $M(A)u^2$ dense in $M(A)u$. For related results we refer the reader to the works [1], [11, Chapter 4], [12,16,17].

2. Closed unital ideals of the algebra $B(G)$

In this section our aim is to prove that the unital closed ideals of $B(G)$ (for G amenable) are exactly the principal ones. The terminology and notation we use are quite standard and they are essentially those introduced in the preceding section. Some other pieces of terminology will be introduced later on when and where needed. We start with a couple of preliminary results. The first lemma is known; the equivalence of (a) to (b) is proved in [1], and the equivalence of (b) to (c) is proved in [16]. For the sake of completeness we include a proof of it, which is quite short.

Lemma 2.1. *Let A be a commutative semisimple Banach algebra with a BAI and $T : A \to A$ be a multiplier. Then the following three assertions are equivalent*:

(a) *T factors as a product of an invertible multiplier and an idempotent multiplier.*
(b) *The ideal $T(A)$ is closed in A and has a BAI.*
(c) *The ideal $T \circ M(A) = \{T \circ S : S \in M(A)\}$ is closed in $M(A)$ and has a unit element.*

Proof. Implication (a) $\Rightarrow$ (b) is obvious since an idempotent multiplier is a homomorphism. To prove implication (b) $\Rightarrow$ (c), suppose that (b) holds. Then observe that, by Cohen's Factorization Theorem [9, 32.26], $A = AA$ and $T(A) = T(A)T(A)$. Hence $T^2(A) = T(A)$. So, given any $a \in A$, there is a $b \in A$ such that $T^2(b) = T(a)$. This shows that $a - T(b)$ belongs to $Ker(T)$. On the other hand, since A is semisimple, $T(A) \cap Ker(T) = \{0\}$. Then the decomposition $a = T(b) + (a - T(b))$ shows that A is the direct sum of the closed ideals $T(A)$ and $Ker(T)$. Since both $T(A)$ and $Ker(T)$ are ideals, every bounded projection θ from A onto $T(A)$ is an idempotent multiplier. It is clear that for such a θ, we have $T \circ \theta = T$. Hence we have the factorization of $T = \theta \circ (T - I + \theta)$, where θ is an idempotent multiplier and $R = T - I + \theta$ an invertible one since its Gelfand transform does not vanish on the Gelfand spectrum of $M(A)$.

Finally to prove implication (c) $\Rightarrow$ (a), suppose that $T \circ M(A)$ is closed in $M(A)$ and has a unit element, say $\theta = T \circ U$. Then θ is an idempotent and, since $T = T \circ I$ is in $T \circ M(A)$, $\theta \circ T = T$. Hence $T = \theta \circ (T + I - \theta)$, with $R = T + I - \theta$ invertible. $\square$

As this lemma makes clear, the main difficulty in proving the analogue of the Glicksberg–Host–Parreau Theorem for the Fourier algebra $A(G)$ (or for an abstract

Banach algebra A) lies in proving that the ideal $T \circ M(A)$ has a unit element as soon as it is closed. It is our purpose in this paper to prove that, under the appropriate hypotheses, the ideal $T \circ M(A)$ has a unit element whenever it is closed. To this end we need some preliminary results with which we proceed. The next couple of results are purely Banach space theoretical and are simple applications of the Hahn–Banach Theorem.

Lemma 2.2. *Let X be a Banach space and $L : X \to X$ be a continuous linear operator with closed range. Then $Ker(L^{**}) = \overline{Ker(L)}^{*}$, where $\overline{Ker(L)}^{*}$ is the weak-star closure of $Ker(L)$ in the space X^{**}.*

Proof. The space $Ker(L^{**})$ being weak-star closed in X^{**}, the inclusion $\overline{Ker(L)}^{*} \subseteq Ker(L^{**})$ is obvious. To prove the reverse inclusion, suppose that we have an element u in $Ker(L^{**})$ which is not in $\overline{Ker(L)}^{*}$. Then, by the Hahn–Banach Theorem, there is a functional $f \in X^*$ such that $\langle f, u \rangle \neq 0$ and f vanishes on $Ker(L)$, i.e. $f \in Ker(L)^{\perp}$. As $L(X)$ is closed in X, $L^*(X^*)$ is weak-star closed in X^*. This implies (again by the Hahn–Banach Theorem) that $Ker(L)^{\perp} = L^*(X^*)$. Hence $f = L^*(g)$ for some $g \in X^*$. This implies that $\langle f, u \rangle = \langle g, L^{**}(u) \rangle = 0$. This contradiction proves the equality $Ker(L^{**}) = \overline{Ker(L)}^{*}$. $\square$

Proposition 2.3. *Let X be a Banach space and $L : X \to X$ be a continuous linear operator with closed range. Then the following assertions are equivalent*:

(a) $Ker(L^2) = Ker(L)$.
(b) $\overline{L^*(X^*) + Ker(L^*)} = X^*$.
(c) $\overline{L^*(L^*(X^*))} = L^*(X^*)$.

Proof. (a) $\Rightarrow$ (b): Suppose that $Ker(L^2) = Ker(L)$. Then, by Lemma 2.2, $Ker(L^{**2}) = Ker(L^{**})$. Now suppose, to get a contradiction, that there exists a functional $f \in X^*$ which is not in the space $\overline{L^*(X^*) + Ker(L^*)}$. Then there exists an element $u \in X^{**}$ such that $\langle u, f \rangle \neq 0$ and u vanishes both on $L^*(X^*)$ and $Ker(L^*)$. Since u vanishes on $L^*(X^*)$, $L^{**}(u) = 0$. Since u also vanishes on $Ker(L^*)$ and since $Ker(L^*)^{\perp} = L^{**}(X^{**})$, $u = L^{**}(v)$ for some $v \in X^{**}$. Here we use the fact that $L^{**}(X^{**})$ is weak-star closed in X^{**} to affirm that $Ker(L^*)^{\perp} = L^{**}(X^{**})$. Hence $L^{**}(L^{**}(v)) = L^{**}(u) = 0$. So, since $Ker(L^{**2}) = Ker(L^{**})$, $L^{**}(v) = 0$. But then

$$\langle u, f \rangle = \langle L^{**}(v), f \rangle = 0.$$

This contradiction proves implication (a) $\Rightarrow$ (b).

To prove implication (b) $\Rightarrow$ (c), it is enough to observe that

$$L^*(X^*) = L^*(\overline{L^*(X^*) + Ker(L^*)}) \subseteq \overline{L^*(L^*(X^*) + Ker(L^*))} = \overline{L^*(L^*(X^*)}.$$

As $L^*(X^*)$ is closed in X^*, we conclude that $\overline{L^*(L^*(X^*))} = L^*(X^*)$.

Finally, to prove implication (c) $\Rightarrow$ (a), since always $Ker(L) \subseteq Ker(L^2)$, it is enough to show that $Ker(L^2) \subseteq Ker(L)$. Let $x \in Ker(L^2)$ be a point. If $x \notin Ker(L)$ then, for some $f \in Ker(L)^{\perp}$, we must have $\langle f, x \rangle \neq 0$. Since $Ker(L)^{\perp} = L^*(X^*)$, $f = L^*(g)$ for some $g \in X^*$. As $\overline{L^*(L^*(X^*))} = L^*(X^*)$, $f = \lim_{n\to\infty} L^*(L^*(g_n))$ for some sequence $(g_n)_{n\in N}$ in X^*. But then

$$\langle f, x \rangle = \lim_{n\to\infty} \langle L^*(L^*(g_n)), x \rangle = \lim_{n\to\infty} \langle g_n, L^2(x) \rangle = 0.$$

This contradiction completes the proof. $\square$

Now let B be a commutative semisimple unital Banach algebra, $\Delta(B)$ be its Gelfand spectrum and $a \in B$ be a given element for which Ba is closed in B. Let $\widehat{a}$ be the Gelfand transform of a and $\Delta(a) = \{f \in \Delta(B) : \widehat{a}(f) \neq 0\}$. Put $\delta(a) = \inf\{|\widehat{a}(f)| : f \in \Delta(a)\}$. This quantity need not be strictly positive. Actually, it is strictly positive iff the set $\Delta(a)$, which is open in $\Delta(B)$, is compact. If this is the case, by the Shilov idempotent Theorem, the ideal Ba is unital. At this point we remark that, as the example (disk algebra) at the end of the preceding section shows, the closedness of the ideal Ba in B does not imply that $\delta(a) > 0$. The next result is the keystone of this paper.

Theorem 2.4. *Let B be a commutative semisimple unital Banach algebra and $a \in B$ be a given element for which the ideal Ba is closed in B. Then $\delta(a) > 0$ iff Ba^2 is dense in Ba.*

Proof. If $\delta(a) > 0$ then $\Delta(a)$, which is the Gelfand spectrum of Ba, is compact and the ideal Ba is unital so that $Ba = BaBa = Ba^2$. So we prove the reverse implication. Suppose then that Ba^2 is dense in Ba. Let $T : B \to B$ be the multiplier defined by $T(x) = ax$. By hypothesis, the range of T is closed. Let $\theta : B \to B/Ker(T)$ be the natural surjection and $\widetilde{T} : B/Ker(T) \to B$ be the injection induced by T so that $\widetilde{T} \circ \theta = T$. The linear operator $\widetilde{T}$ is an isomorphism from $B/Ker(T)$ onto $T(B)$. Then the adjoint $\widetilde{T}^* : T(B)^* \to (B/Ker(T))^*$ of $\widetilde{T}$ is also an isomorphism. As $T(B)^* = B^*/T(B)^{\perp}$ and $(B/Ker(T))^* = Ker(T)^{\perp} = T^*(B^*)$, we consider $\widetilde{T}^*$ as an isomorphism from $B^*/T(B)^{\perp}$ onto $T^*(B^*)$. Hence there exist two constants $\alpha > 0$ and $\beta > 0$ such that, for $f \in B^*$,

$$\alpha . ||f + T(B)^{\perp}|| \leqslant ||\widetilde{T}^*(f + T(B)^{\perp})|| \leqslant \beta . ||f + T(B)^{\perp}||.$$

Since $T(B)^{\perp} = Ker(T^*)$, $\widetilde{T}^*(f + T(B)^{\perp}) = \widetilde{T}^*(f + Ker(T^*)) = T^*(f)$. Since for any $f \in B^*$,

$$||f + T(B)^{\perp}|| = d(f, Ker(T^*)),$$

here $d(f, Ker(T^*))$ denotes the distance of f to the subspace $Ker(T^*)$, the above inequalities become

$$\alpha \,.\, d(f, Ker(T^*)) \leqslant ||T^*(f)|| \leqslant \beta \,.\, d(f, Ker(T^*)).$$

Now let $f \in \Delta(a)$. Then $T^*(f) = \widehat{a}(f)f$. As B is unital, the norm of f is one so that

$$||T^*(f)|| = |\widehat{a}(f)|.$$

Hence, for $f \in \Delta(a)$, we have

$$\alpha \,.\, d(f, Ker(T^*)) \leqslant |\widehat{a}(f)| \leqslant \beta \,.\, d(f, Ker(T^*)).$$

From these inequalities we conclude that

$$\delta(a) > 0 \quad \text{iff} \quad d(\Delta(a), Ker(T^*)) > 0.$$

So far we did not use the hypothesis that the ideal $Ba^2 = T(T(B))$ is dense in $Ba = T(B)$; now it is time to use it. Exactly as in the proof of the preceding proposition, a simple Hahn–Banach argument will show that, $T(T(B))$ is dense in $T(B)$ iff $T(B) + Ker(T)$ is dense in B. It is in this latter form that we shall use it. For a contradiction, assume that $\delta(a) = 0$. Then $d(\Delta(a), Ker(T^*)) = 0$. So there exist a sequence $(f_n)_{n \in N}$ in $\Delta(a)$ and a sequence $(g_n)_{n \in N}$ in $Ker(T^*)$ such that $||f_n - g_n|| \to 0$, as $n \to \infty$. Since the norm of each f_n is one, the sequence $(g_n)_{n \in N}$ is bounded. The Gelfand spectrum of B being compact in the weak-star topology of B^*, the sequence $(f_n)_{n \in N}$ has a subnet $(f_{n_\alpha})_{\alpha \in I}$ that converges in the weak-star topology of B^* to an element f of $\Delta(B)$. If necessary, passing still to another subnet, we can suppose that the net $(g_{n_\alpha})_{\alpha \in I}$ also converges in the weak-star topology of B^* to an element g of $Ker(T^*)$. Now, by the lower semicontinuity of the norm of B^* with respect to the weak-star topology of B^*, we have

$$||f - g|| \leqslant \liminf ||f_{n_\alpha} - g_{n_\alpha}|| = 0.$$

Hence $f = g$, and $f \in Ker(T^*)$. So f vanishes on $T(B) = Ba$. On the other hand, for any $x \in Ker(T)$ and any $g \in \Delta(a)$, $\langle g, x \rangle = 0$. Hence, since $f_{n_\alpha} \in \Delta(a)$ and $f_{n_\alpha} \to f$, f also vanishes on $Ker(T)$. Then, since $\overline{T(B) + Ker(T)} = B$, $f = 0$ on the whole of B, which is not possible since $f \in \Delta(B)$. This contradiction proves that $\delta(a) > 0$, and so Ba has a unit element. $\square$

The next theorem is our first main result.

Theorem 2.5. *Let X be a Banach space and $L : X \to X$ be a continuous linear operator whose range is closed. Suppose that $B = X^*$ is a commutative semisimple unital Banach algebra for some multiplication and that $T = L^*$ is a multiplier on B. Then the ideal $T(B)$ has a unit element iff $Ker(L^2) = Ker(L)$.*

Proof. First observe that, B being a unital Banach algebra and L^* being a multiplier on B, for some $a \in B$, $L^*(x) = ax$ so that $L^*(B) = Ba$. Now suppose that $Ker(L^2) = Ker(L)$. This, by Proposition 2.3, implies that $\overline{L^*(L^*(X^*))} = L^*(X^*)$ so that Ba^2 is dense in Ba. Then, by the preceding theorem, the ideal $T(B) = Ba$ has a unit element.

Conversely, suppose that $T(B) = Ba$ has a unit element, say u. Then $Ba = Bu$ and $Ker(T) = B(1-u)$, where 1 is the unit element of B. Hence $T(B) + Ker(T) = L^*(X^*) + Ker(L^*) = X^*$, and so $Ker(L^2) = Ker(L)$ by Proposition 2.3 again. $\square$

Now we are in a position to prove the result announced in the abstract. To this end, let G be an arbitrary locally compact group, $B(G)$ be its Fourier–Stieltjes algebra and $A(G)$ be its Fourier algebra. The algebra $B(G)$ is a commutative semisimple unital Banach algebra and $A(G)$ is a closed ideal of it. Moreover $B(G)$ is the dual of the group C^*-algebra $C^*(G)$ and that, for $u \in B(G)$ fixed, the multiplication operator $v \mapsto vu$ is continuous in the weak-star topology of $B(G)$. We also recall that the algebra $A(G)$ is a commutative, regular semisimple Tauberian Banach function algebra. The Gelfand spectrum of $A(G)$, via Dirac measures, is G. For these results we refer the reader to the paper [4] of Eymard. Moreover, the algebra $A(G)$ has a BAI iff the group G is amenable [15, Theorem 10.4]. In this case, $B(G)$ is the multiplier algebra of $A(G)$ [3] or [15, Proposition 19.11].

Now assume that G is amenable, and fix an element u in $B(G)$ such that the ideal $B(G)u$ is closed in $B(G)$. To the element u we associate the linear operator $L : C^*(G) \to C^*(G)$ defined by $L(f) = uf$. Here, as usual, uf is the functional on $B(G)$ defined by $\langle uf, v \rangle = \langle vu, f \rangle$. The operator L is a continuous linear operator on $C^*(G)$ and $L^*(v) = vu$ so that L^* is a multiplier on $B(G)$ and $L^*(B(G)) = B(G)u$. The preceding theorem shows that $B(G)u$ is unital iff $Ker(L^2) = Ker(L)$. The next result shows that, when G is amenable, the equality $Ker(L^2) = Ker(L)$ holds automatically as a consequence of the fact that the algebra $A(G)$ is Tauberian. See also Lemma 3.3 and Theorem 3.4 below.

Lemma 2.6. *Let G be a locally compact amenable group and u be an element of $B(G)$ such that the ideal $B(G)u$ is closed in $B(G)$. Let $L : C^*(G) \to C^*(G)$ be the linear operator defined by $L(f) = uf$. Then $Ker(L^2) = Ker(L)$.*

Proof. Let $T : A(G) \to A(G)$ be the multiplier defined by $T(a) = au$. It is clear that, for $a \in A(G)$, $T(a) = L^*(a) = au$. As G is amenable, $C^*(G)$ is a subspace of $VN(G) = A(G)^*$ and, for $f \in C^*(G)$, the functional $L(f) = T^*(f) = uf$, being in $C^*(G)$, is in $VN(G)$. Hence, for an $f \in C^*(G)$, to show that $uf = 0$ as an element of $C^*(G)$ it is enough to show that $uf = 0$ as a functional on $A(G)$. Now, since $B(G)u$ is closed in $B(G)$, by Lemma 3.1 below, the ideal $A(G)u$ is closed in $A(G)$. Let

$F = \{g \in G : a(g) = 0\}$ and $O = G \backslash F$. The set F is the hull of the ideal $A(G)u$ and the set O is its Gelfand spectrum.

Step 1: In this step our aim is to show that the set O, which is open in G, is also closed in G. To prove that O is closed in G, the following is the main property we need:

$$\forall f \in O \quad \exists a \in A(G) \text{ such that } a(f) = 1, \ ||a|| \leqslant 1 \text{ and}$$
$$\text{the support of } a \text{ is contained in } O.$$

To see that this property holds, let $f \in O$ be a given point. Then let U be a compact neighborhood of the identity element of G such that $fUU^{-1} \cap F = \emptyset$. Let $b = \frac{\chi_U}{\mu(U)}$ and $c = \chi_{fU}$. Here χ_U is the characteristic function of U and μ is the left Haar measure of G. The functions b and c are both in $L^2(G)$ so that $a = c * b^\vee$ (here $b^\vee(g) = b(g^{-1})$) is in $A(G)$, its norm is one, $a(f) = 1$ and the support of a is contained in O.

Now let I be the closure of the ideal

$$\{x \in A(G) : \text{The support of } x \text{ is compact and } x = 0$$
$$\text{on an open set } O_x \text{ containing } F\}.$$

The ideal I is the smallest closed ideal whose hull is F [2, Proposition 4.1.20, p. 414]. Since $A(G)u$ is a closed ideal of $A(G)$ and its hull is F, $I \subseteq A(G)u$. Let $C_c(G)$ be the space of the continuous functions on G with compact support. Now let $(a_n)_{n \in N}$ be a sequence in $A(G) \cap C_c(G)$ that converges to a. Since $A(G)$ is Tauberian, such a sequence exists. Since $aa_n \in I$, we conclude that $a^2 \in T(A(G))$. Hence $a^2 = T(b)$ for some $b \in A(G)$. Since $T(A(G))$ is closed in $A(G)$, by The Open Mapping Theorem, we can assume that the norm of b is less than a certain constant c, which does not depend on a or f. This constant being independent from the f chosen in O and $|b(f)| \leqslant ||b||$, we have

$$1 = a^2(f) = \langle T(b), f \rangle = u(f)b(f) \leqslant |u(f)| \, . \, c$$

so that $\inf\{|u(f)| : f \in O\} \geqslant 1/c$. Hence $\delta(u) = \inf\{|u(f)| : f \in O\} \geqslant 1/c > 0$. This, by continuity of u, implies that O is closed in G.

Step 2: In this step our aim is to show that the ideal $T(A(G)) + Ker(T)$ is dense in $A(G)$. To see this let us first see that O is the hull of the ideal $Ker(T)$. To prove this latter point, the inclusion $O \subseteq Hull(Ker(T))$ being clear, we prove the reverse inclusion. Let, if there is any, f be a point in $Hull(Ker(T)) \backslash O$. Then, the algebra $A(G)$ being regular, there is an element $a \in A(G)$ such that $a(f) = 1$ and $a = 0$ on O. Then $au = 0$ on G so that $a \in Ker(T)$, which is not possible since $f \in Hull(Ker(T))$ and $a(f) = 1$. This contradiction shows that $Hull(Ker(T)) = O$. This implies that the hull of the ideal $T(A(G)) + Ker(T)$ is empty. The algebra $A(G)$ being Tauberian, this in turn implies that the ideal $T(A(G)) + Ker(T)$ is dense in $A(G)$.

Step 3: Finally, let us see that $Ker(L^2) = Ker(L)$. The inclusion $Ker(L) \subseteq Ker(L^2)$ being clear, we prove the reverse inclusion. Let $f \in C^*(G)$ be such that $L(L(f)) = u^2 f = 0$. Then, for any $v \in B(G)$, $\langle uf, vu \rangle = 0$. In particular, for any $a \in A(G)$,

$\langle uf, au \rangle = 0$. Since for any $a \in A(G)$, $\langle uf, au \rangle = 0$, the functional uf vanishes on $T(A(G)) = A(G)u$. Since, for $a \in Ker(T)$, $\langle uf, a \rangle = \langle f, au \rangle = 0$, uf vanishes on $Ker(T)$ too. Hence, the ideal $T(A(G)) + Ker(T)$ being dense in $A(G)$, $uf = 0$ on $A(G)$. Thus $L(f) = 0$, and $Ker(L^2) = Ker(L)$. $\square$

We do not know whether this result holds or not without amenability hypothesis on G. The next result is the noncommutative version of the Glicksberg–Host–Parreau Theorem. The proof of this result, being now immediate, is omitted.

Theorem 2.7. *Let G be a locally compact amenable group and I be a closed ideal of $B(G)$. Then the ideal I has a unit element iff it is a principal ideal.*

3. An abstract form of the Glicksberg–Host–Parreau Theorem

In this section our aim is to present an abstract version of the Glicksberg–Host–Parreau Theorem. This will permit us to see better what lies behind this result.

Let A be a commutative semisimple Banach algebra having a BAI $(e_\alpha)_{\alpha \in I}$. On the second dual of A we put the first Arens multiplication, which is defined in three steps as follows. Let $m, n \in A^{**}$, $f \in A^*$ and $a, b \in A$. Then the elements mn of A^{**}, nf and fa of A^* are defined by

$$\langle fa, b \rangle = \langle f, ab \rangle$$

$$\langle mf, a \rangle = \langle m, fa \rangle,$$

and

$$\langle mn, f \rangle = \langle m, nf \rangle.$$

With this product, A^{**} is a Banach algebra and A is in the center of it. Let e be a weak-star cluster point of the BAI $(e_\alpha)_{\alpha \in I}$. This e is chosen once for all and will be kept fixed throughout this section. This element e is a right identity in A^{**} (i.e. $me = m$ for all $m \in A^{**}$), so an idempotent. Hence $P(m) = em$ is a bounded projection on A^{**}, which induces the decomposition $A^{**} = P(A^{**}) \oplus (I - P(A^{**}))$. This decomposition will be denoted simply as $A^{**} = eA^{**} + (1 - e)A^{**}$. On the other hand, let $A^*A = \{fa : f \in A \text{ and } a \in A\}$. This is a closed subspace of A^* and its dual can be identified with the subspace eA^{**} of A^{**}, see for instance [13, Section 3]. For any $T \in M(A)$, $T^*(fa) = T^*(f)a$ so that $T^*(A^*A) \subseteq T^*(A^*)A$. In particular,

$$\langle T^*(fa), e \rangle = \langle T^*(f), a \rangle = \langle f, T(a) \rangle.$$

The multiplier algebra $M(A)$ embeds continuously into eA^{**} by (see [14])

$$T \mapsto eT^{**}(e).$$

As $||fa|| \leqslant ||f||.||a||$ and

$$\sup_{||fa|| \leqslant 1} |\langle eT^{**}(e), fa \rangle| = \sup_{||fa|| \leqslant 1} |\langle T^*(f), a \rangle| \geqslant ||T||,$$

this embedding of the algebra $M(A)$ into eA^{**} is topological. Now on we regard $M(A)$ as a subspace of eA^{**}, embedded in the indicated way. We assume that there exists a closed subspace X of A^*A, invariant under each T^* $(T \in M(A))$, such that $M(A)$ identifies naturally with X^*. Here the word "naturally" means that in the duality (X, X^*), $T \in X^*$ acts on an element $f = ga$ of X through the formula $T(f) = \langle g, T(a) \rangle$. Hence the quantity $T(f)$ is the same as $\langle f, eT^{**}(e) \rangle$ in (A^*A, eA^{**})-duality so that $T(f) = \langle f, eT^{**}(e) \rangle = \langle f, T \rangle$. So, the notation $\langle f, T \rangle$ for $T(f)$ is unambiguous and $\langle f, T \rangle = T(f)$. One consequence to be noted of the hypothesis that X is invariant under each T^* $(T \in M(A))$ is that in $M(A)$ the multiplication is separately weak-star continuous.

In the rest of the paper, A will be a commutative semisimple Banach algebra with a BAI and the letter X will have the meaning assigned to it in the preceding paragraph.

The following classical Banach algebras fit perfectly into the abstract scheme described above.

(1) $A = L^1(G)$, the group algebra of a locally compact abelian group G, with $X = C_0(G)$, the space of the continuous functions on G that vanish at infinity.

(2) $A = A(G)$, the Fourier algebra of a locally compact amenable group G, with $X = C^*(G)$, the group C^*-algebra of G [4].

(3) $A = A_p(G)$ $(1 < p < \infty)$, the Herz–Figa–Talamanca algebra of a locally compact amenable group G, with $X = PF_p(G)$, the space of the pseudo-functions on G [8].

These are not the only Banach algebras that fit in the scheme described above. For instance, any commutative semisimple Banach algebra A with a BAI which is an ideal in its second dual, with $X = A^*A$, also fits into this scheme.

Finally at this point, we recall that the algebra $A_p(G)$ is a commutative semisimple regular Tauberian Banach algebra; it has a BAI iff G is amenable and that, for $p = 2$, $A_p(G) = A(G)$. Moreover, if G is abelian, $A(G) = L^1(\widehat{G})$ (via Fourier transform), where $\widehat{G}$ is the dual group of G. The Gelfand spectrum of $A_p(G)$, via Dirac measures, is G; and, when G is amenable, the multiplier algebra of $A_p(G)$ is $B_p(G)$, the space of p-pseudomeasures on G [15, Section 19].

The next result is aimed to clarify, for a multiplier $T \in M(A)$, the connections between the closedness of the spaces $T \circ M(A)$, $T(A)$ and $L(X)$, where $L : X \to X$ is the linear operator associated to the multiplier T through the formula $L(f) = T^*(f)$.

Lemma 3.1. *Let $T : A \to A$ be a multiplier on A and $L : X \to X$ be the linear operator associated to T through the formula $L(f) = T^*(f)$. Then the following equivalences hold*:

(a) *The ideal $T \circ M(A)$ is closed in $M(A)$ iff the range $L(X)$ of L is closed in X.*
(b) *The ideal $T \circ M(A)$ is closed in $M(A)$ iff the ideal $T(A)$ is closed in A.*

Proof. (a) To prove this equivalence, it is enough to see what $L^* : M(A) \to M(A)$ is. Indeed, for $S \in M(A)$, and $f \in X$,

$$\langle L^*(S), f \rangle = \langle S, L(f) \rangle = \langle S, T^*(f) \rangle = \langle TS, f \rangle.$$

More precisely,

$$\langle L^*(S), f \rangle = \langle T^*(f), eS^{**}(e) \rangle = \langle f, T^{**}(e(S^{**}(e)) \rangle$$
$$= \langle f, eT^{**}(S^{**}(e)) \rangle = \langle f, e(T \circ S)^{**}(e) \rangle = \langle f, T \circ S \rangle$$

so that $L^*(S) = TS$. Hence the ideal $T \circ M(A)$ is closed in $M(A)$ iff the range $L(X)$ of L is closed in X. Observe that when $T \circ M(A)$ is closed in $M(A)$ it is in fact weak-star closed.

(b) To prove the second equivalence, first suppose that $T \circ M(A)$ is closed in $M(A)$. Let, for $a \in A$, $\tau_a : A \to A$ be the multiplier defined by $\tau_a(x) = ax$. Since A has a BAI, the mapping $a \mapsto \tau_a$ is an isomorphism from A onto a closed ideal of $M(A)$. Now let $H = \{S \in M(A) : T \circ S = 0\}$. This is a closed ideal of $M(A)$ and, for $x \in Ker(T)$, $\tau_x \in H$. Now define the mapping

$$j : A/Ker(T) \to M(A)/H$$

by $j(\tau_a + Ker(T)) = \tau_a + H$. As $||\tau_a + H|| \leqslant ||a + Ker(T)||$, j is continuous. Now remark that, for any $S \in H$ and $a \in A$, $S(a) \in Ker(T)$. Hence, since A has a BAI $(e_\alpha)_{\alpha \in I}$, for any $a \in A$ and $S \in H$, we have

$$||\tau_a + S|| = \sup\{||\tau_a(x) + S(x)|| : ||x|| \leqslant 1\} \geqslant 1/\beta ||ae_\alpha + S(e_\alpha)||$$
$$\geqslant 1/\beta ||ae_\alpha + Ker(T)|| \geqslant 1/2\beta ||a + Ker(T)||,$$

where β is a constant such that $||e_\alpha/\beta|| \leqslant 1$ for each $\alpha \in I$. From these inequalities we see that j is an isomorphism, and so its range is closed in $M(A)/H$. On the other hand, since the ideal $T \circ M(A)$ is closed in $M(A)$, the mapping $F : M(A)/H \to T \circ M(A)$ defined by $F(S + H) = T \circ S$ is an onto isomorphism. Hence $F(j(A/Ker(T))$ is closed, so a Banach subspace of $T \circ M(A)$. As, for $a \in A$, $F(j(a + Ker(T)) = T \circ \tau_a$, we conclude that $T(A)$ is a Banach space, so closed in A.

Conversely, suppose that $T(A)$ is closed in A. Let $(T \circ S_n)_{n \in N}$ be a sequence in $T \circ M(A)$ that converges to some multiplier R in $M(A)$. Then, for any $x \in A$, $T(S_n(x)) \to R(x)$ in A. Hence $R(x) \in T(A)$ so that $R(A) \subseteq T(A)$. Hence every $R(a)$ is a $T(b)$ for some $b \in A$. Now we define a mapping $\theta : A \to A/Ker(T)$ as follows:

$$\theta(a) = b + ker(T),$$

where b is such that $R(a) = T(b)$. This is a well-defined linear mapping. Let us see that it is continuous. Suppose that, for some sequence $(a_n)_{n \in N}$ in A,

$$a_n \to a \quad \text{and} \quad \theta(a_n) = b_n + ker(T) \to b + Ker(T),$$

where b_n is such that $R(a_n) = T(b_n)$. As R is continuous, $R(a_n) \to R(a)$. Since $b_n + Ker(T) \to b + Ker(T)$, and since the mapping $\widetilde{T} : A/Ker(T) \to T(A)$, defined by $\widetilde{T}(x + Ker(T)) = T(x)$, is continuous, $T(b_n) \to T(b)$ too. As $R(a_n) = T(b_n)$ and $R(a_n) \to R(a)$, we see that $R(a) = T(b)$. Then $\theta(a) = b + Ker(T)$ so that, by the Closed Graph Theorem, θ is continuous. Hence its adjoint $\theta^* : T^*(A^*) \to A^*$ is also continuous. To see what θ^* is, let $T^*(f)$ be an element of $T^*(A^*)$. Then, for $a \in A$,

$$\langle \theta^*(T^*(f)), a \rangle = \langle T^*(f), \theta(a) \rangle = \langle f, T(b) \rangle,$$

where b is such that $R(a) = T(b)$. Hence $\theta^*(T^*(f)) = R^*(f)$. Next define a functional $\phi : T^*(X) \to C$ by

$$\phi(T^*(f)) = \langle R, f \rangle.$$

Since $T^*(X) \subseteq T^*(A^*)A$ and, since for any ga in A^*A, $\langle ga, e \rangle = \langle g, a \rangle$, from the equality $\theta^*(T^*(f)) = R^*(f)$, we get that

$$\begin{aligned} |\phi(T^*(f))| &= |\langle R, f \rangle| = |\langle f, eR^{**}(e) \rangle| = |\langle R^*(f), e \rangle| \\ &= |\langle \theta^*(T^*(f)), e \rangle| = |\langle T^*(f), \theta^{**}(e) \rangle| \leqslant \|T^*(f)\|.\|\theta^{**}(e)\|. \end{aligned}$$

Hence ϕ is a continuous functional on the subspace $T^*(X)$ of X. Let S be a Hahn–Banach extension of it to the space X so that $S \in M(A)$ and, for any $f \in X$, $\langle S, T^*(f) \rangle = \langle R, f \rangle$. Hence $R = T \circ S$, and the ideal $T \circ M(A)$ is closed in $M(A)$. $\square$

Now let $T : A \to A$ be a multiplier with closed range. Associate to T the linear operator $L : X \to X$ defined through the formula $L(f) = T^*(f)$. Then, by the preceding lemma, $L(X)$ is closed in X and $L^*(M(A)) = T \circ M(A)$ is closed in $M(A)$. Hence, by Theorem 2.5, the ideal $T \circ M(A)$ is unital iff $Ker(L^2) = Ker(L)$. In deciding when $Ker(L^2) = Ker(L)$, the key observation is the following.

Lemma 3.2. *Let $T : A \to A$ be a multiplier with closed range and $L : X \to X$ be the linear operator associated to it through the formula $L(f) = T^*(f)$. Then $Ker(L^2) = Ker(L)$ iff $\overline{T(A) + Ker(T)} = A$.*

Proof. Indeed, suppose first that $Ker(L^2) = Ker(L)$. Then, by Theorem 2.5 and Lemma 2.1, T factors as a product of an invertible multiplier R and an idempotent multiplier θ. Then $T(A) = \theta(A)$ and $Ker(T) = Ker(\theta)$. Hence, θ being a projection, $T(A) + Ker(T) = \theta(A) + Ker(\theta) = A$. Conversely, suppose that $\overline{T(A) + Ker(T)} = A$. Let $f \in X$ be such that $L^2(f) = T^*(T^*(f)) = 0$. Then, clearly $T^*(f)$ vanishes on the subspace $T(A) + Ker(T)$ of A. This subspace being dense in A, we conclude that $L(f) = T^*(f) = 0$. Hence $Ker(L^2) \subseteq Ker(L)$. The reverse inclusion being always true, $Ker(L^2) = Ker(L)$. $\square$

In the following results our concerning will be the density of the ideal $T(A) + Ker(T)$ in the algebra A. To this end the following notion, which is a weak form of the Urysohn Lemma adapted to the algebra A, has turned out to be quite useful [16]:

An open subset O of $\Delta(A)$ will be called a "u-set with respect to the algebra A" if there is a constant $c>0$ such that, for each $f\in O$, there is an element $a\in A$ with $\widehat{a}(f) = 1$, $||a||\leqslant c$ and the support of $\widehat{a}$ is contained in the set O.

Lemma 3.3. *For any locally compact group G and $1<p<\infty$, every open subset O of G is a u-set with respect to the algebra $A_p(G)$.*

Proof. Let O be an open subset of G and f be an element of O. Let $F = G\backslash O$ and, as in the proof of Lemma 2.6, let U be a compact neighborhood of the identity element of G such that $fUU^{-1}\cap F = \emptyset$. Let $b = \frac{\chi_U}{\mu(U)}$ and $c = \chi_{fU}$. Here χ_U is the characteristic function of U and μ is the left Haar measure of G. The function c is in $L^p(G)$ and the function b in $L^q(G)$ $(1/p + 1/q = 1)$ so that $a = c * b^\vee$ (here $b^\vee(g) = b(g^{-1})$) is in $A_p(G)$; its norm is one, $a(f) = 1$ and the support of a is contained in O. Hence O is a u-set with respect to the algebra $A_p(G)$. $\square$

The Gelfand spectrum $\Delta(M(A))$ of the algebra $M(A)$ is the union of the set $\Delta(A)$ and the set $H(A)$, the hull of A, where A is considered as an ideal in $M(A)$ [10]. The set $\Delta(A)$ is open in $M(A)$ but is far from being dense in it (Wiener–Pitt phenomenon). For a multiplier T on A, by $\widehat{T} : \Delta(M(A))\to C$ we denote its Gelfand transform. Let $\Delta(T) = \{f\in\Delta(A) : \widehat{T}(f)\neq 0\}$ and $\delta(T) = \inf\{|\widehat{T}(f)| : f\in\Delta(T)\}$.

Theorem 3.4. *Suppose that the algebra A is also regular and Tauberian. Then, for any multiplier $T : A\to A$ with a closed range, the following three assertions are equivalent*:

(a) *$T(A) + Ker(T)$ is dense in A.*
(b) $\delta(T)>0$.
(c) *The set $\Delta(T)$ is a u-set with respect to the algebra A.*

Proof. (a) $\Rightarrow$ (b): Suppose that $T(A) + Ker(T)$ is dense in A. Then, by Lemma 3.2, T factors as a product of an invertible multiplier S and an idempotent multiplier θ. Then $\Delta(T) = \Delta(\theta)$ and, for $f\in\Delta(T)$, $\widehat{T}(f) = \widehat{S}(f)$. Since S is invertible, $\inf\{|\widehat{S}(f)| : f\in\Delta(M(A))\}>0$. From this we conclude that

$$\delta(T) = \inf\{|\widehat{T}(f)| : f\in\Delta(T)\}>0.$$

Implication (b) $\Rightarrow$ (c) is proved in [16, Theorem 2.12]. To prove implication (c) $\Rightarrow$ (a), we first remark that, as one can see it immediately, $\Delta(T)$ is the Gelfand spectrum of the ideal $T(A)$ and $\Delta_0(T) = \{f\in\Delta(A) : \widehat{T}(f) = 0\}$ is its hull. Now suppose that $\Delta(T)$ is a u-set with respect to the algebra A. So there is a constant $c>0$ such that, for each $f\in\Delta(T)$, there is an element $a\in A$ for which $||a||\leqslant c$, $\langle a,f\rangle = 1$ and the support of $\hat{a}$ is contained in $\Delta(T)$. Fix an $f\in\Delta(T)$ and let $a\in A$ be such that $||a||\leqslant c$,

$Supp(\widehat{a}) \subseteq \Delta(T)$ and $\langle a,f \rangle = 1$. Let I be the closure of the ideal

$$\{x \in A : Supp(\hat{x}) \text{ is compact and } \widehat{x} \text{ is null on an open set } O_x \text{ containing } \Delta_0(T)\}.$$

Then, since the hull of the closed ideal $T(A)$ is $\Delta_0(T)$, $I \subseteq T(A)$ [2, Proposition 4.1.20, p. 414]. Let $(a_n)_{n \in N}$ be a sequence in A such that the support of each $\widehat{a_n}$ is compact and $||a_n - a|| \to 0$. As the support of $\hat{a}$ is contained in $\Delta(T)$, $\widehat{a}$ is null on an open set O_a containing $\Delta_0(T)$ so that, for each $n \in N$, the product aa_n is in the ideal I, so in $T(A)$. Hence a^2 is in $T(A)$ so that $a^2 = T(b)$ for some $b \in A$. Since $T(A)$ is closed in A, by The Open Mapping Theorem, we can assume that the norm of b is less that a certain constant c_1, which depends on c but not on a or f. This constant being independent from the f chosen in $\Delta(T)$,

$$1 = \langle a^2,f \rangle = \langle T(b),f \rangle = \widehat{T}(f)\langle b,f \rangle \leqslant |\widehat{T}(f)| \,.\, c_1$$

so that $\inf\{|\widehat{T}(f)| : f \in \Delta(T)\} \geqslant 1/c_1$. Hence $\delta(T) > 0$. In particular, the set $\Delta(T)$, which is open in $\Delta(A)$, is closed in $\Delta(A)$. Hence, since A is regular, Tauberian and has a BAI, the sets $\Delta(T)$ and $\Delta_0(T)$ are sets of synthesis so that $T(A) = ker(\Delta_0(T))$. Since the hull of the ideal $T^2(A)$ is also $\Delta_0(T)$, the ideal $T^2(A)$ is dense in $T(A)$. Hence, by Proposition 2.2, the ideal $T(A) + Ker(T)$ is dense in A. $\square$

Theorem 3.4 together with Lemma 3.3 imply the following result.

Lemma 3.5. *Let G be a locally compact amenable group, $1 < p < \infty$ and $A = A_p(G)$. Then, for any multiplier T with closed range on A, $T(A) + Ker(T)$ is dense in A.*

Another large class of Banach algebras A for which the ideal $T(A) + Ker(T)$ is dense in A whenever $T(A)$ is closed in A is given in the next result.

Proposition 3.6. *Suppose that the spectrum $\Delta(A)$ of the algebra is discrete and A is Tauberian. Then for any multiplier $T : A \to A$ with closed range, the ideal $T(A) + Ker(T)$ is dense in A.*

Proof. First we observe that, $\Delta(A)$ being discrete, the algebra A is regular. For any $f \in \Delta(A)$, by Shilov Idempotent Theorem, there is an element a such that $\widehat{a}$ is the characteristic function of the one-point set $\{f\}$. Then, as one can see immediately, $T(a) = \widehat{T}(f)a$. Hence $a \in T(A)$ if $\widehat{T}(f) \neq 0$; and $a \in Ker(T)$ if $\widehat{T}(f) = 0$. So, a belongs to the ideal $T(A) + Ker(T)$. This implies that the set $A_c = \{a \in A : Supp(\widehat{a})$ is compact$\}$ is contained in the ideal $T(A) + Ker(T)$. Hence, the algebra A being regular and Tauberian, we conclude that $T(A) + Ker(T)$ is dense in A. $\square$

The next result, which is our second main result, is an abstract analogue of the Glicksberg–Host–Parreau Theorem. We state this result under its full hypotheses.

Theorem 3.7. *Let A be a commutative semisimple regular Tauberian Banach algebra with a BAI. Suppose that every open subset of $\Delta(A)$ is a u-set with respect to A and that the multiplier algebra $M(A)$ of A can be identified in a natural way with the dual of a closed subspace X of A^*A which is invariant under each T^* $(T \in M(A))$. Then*

(a) *A closed ideal of $M(A)$ is unital iff it is a principal ideal.*
(b) *The range of a multiplier T is closed in A iff T is the product of an invertible multiplier and an idempotent multiplier.*

Proof. We just sketch the proof of this theorem, which is almost immediate. Suppose first that, for some $T \in M(A)$, the ideal $T \circ M(A)$ is closed in $M(A)$. The multiplication in $M(A)$ being weak-star continuous, $T \circ M(A) = L^*(M(A))$ for some continuous linear operator $L : X \to X$. This linear operator is given by $L(f) = T^*(f)$ on the subspace X of A^*A. By Theorem 3.4, $T(A) + Ker(T)$ is dense in A. Hence, by Proposition 2.3, $Ker(L^2) = Ker(L)$. Then, by Theorem 2.5, $T \circ M(A)$ has a unit element. The other implication being trivial, assertion (a) is proved. Assertion (b) follows now from Lemmas 2.1 and 3.1. $\square$

The proof of the next result, which is an immediate application of the preceding theorem and Lemma 3.5, is omitted.

Corollary 3.8. *Let G be a locally compact amenable group, $1 < p < \infty$ and $u \in B_p(G)$ be a given element. Then the ideal $A_p(G)u$ is closed in $A_p(G)$ iff u factors as $u = b\theta$, where $b \in B_p(G)$ is invertible and $\theta \in B_p(G)$ is idempotent.*

In the case where $\Delta(A)$ is discrete we have a fairly complete information about the ideals of the algebra A that possess a BAI.

Theorem 3.9. *Suppose that the algebra A is Tauberian and its spectrum $\Delta(A)$ is discrete. Then, for a closed ideal I of A, the following assertions are equivalent:*

(a) *The ideal I has a BAI.*
(b) *There is a multiplier $T : A \to A$ such that $I = T(A)$.*
(c) *There is an idempotent multiplier $\theta : A \to A$ such that $I = \theta(A)$.*
(d) *There is a closed ideal J in A such that $I \oplus J = A$.*

Proof. Let $A_c = \{a \in A : Supp(\widehat{a}) \text{ is compact}\}$ and, for each $a \in A$, $\tau_a : A \to A$ be the multiplication operator defined by $\tau_a(x) = ax$. For $a \in A_c$, τ_a is a finite-rank operator, so it is compact. Hence, A being Tauberian, for each $a \in A$, τ_a is compact so that A is an ideal in its second dual A^{**}. Hence the multiplier algebra $M(A)$ of A can be identified with the subspace eA^{**} of A^{**}. This being observed, to prove that (a) implies (b), let I be a closed ideal in A with a BAI $(d_\alpha)_{a \in I}$. Let E be the hull of the ideal I and u be a weak-star cluster point of the net $(d_\alpha)_{\alpha \in I}$ in A^{**}. Then $u \in I^{**}$, $\langle u, f \rangle = 0$ for each $f \in E$, and $\langle u, f \rangle = 1$ for each $f \in \Delta(I)$. Let $T : A \to A$ be the multiplier defined by $T(a) = au$. Then T is an idempotent multiplier and, by semisimplicity of A, $T(A) = Au = I$. To prove implication (b) $\Rightarrow$ (c), suppose that

(b) holds. Then, by Proposition 3.6, $T(A) + Ker(T)$ is dense in A. Hence, by Lemma 3.2, $Ker(L^2) = Ker(L)$. Here $L : A^*A \to A^*A$ is the mapping defined by $L(f) = T^*(f)$. Hence, by Lemma 3.1 and Theorem 2.5, T factors as a product of an idempotent multiplier $\theta \in M(A)$ and an invertible multiplier $S \in M(A)$. So $I = T(A) = \theta(A)$. Implications (c) $\Rightarrow$ (d) and (d) $\Rightarrow$ (a) are immediate. $\square$

As a simple application of the preceding theorem, consider the space c_0 of the complex sequences that converge to zero. Let $A = c_0 \widehat{\otimes} c_0$ be the projective tensor product algebra of c_0 by itself. This is not a C^*-algebra of course. This algebra satisfies all the hypothesis of the preceding theorem. Hence every closed ideal I of A with a BAI is complemented in A.

We finish the paper with the following observation. Let A be as in Theorem 3.7 above, and let us say that a complemented ideal I of A is "invariantly complemented" if $I = \theta(A)$ for some idempotent multiplier $\theta \in M(A)$. Then, since the multiplier algebra of $M(A)$ is itself, $\theta \circ M(A)$ is also an invariantly complemented ideal in $M(A)$. Since A has a BAI, A is weak-star dense in $M(A)$. This implies that the ideal $\theta(A)$ is weak-star dense in the ideal $\theta \circ M(A)$. Conversely, by Theorem 3.7, since every closed unital ideal of $M(A)$ is of the form $\theta \circ M(A)$ for some idempotent multiplier $\theta \in M(A)$, we see that there exists a bijective correspondence between the set of the invariantly complemented ideals of A and the set of the invariantly complemented ideals of $M(A)$. In the case where $A = A(G)$ is the Fourier algebra of an amenable group G, there also exists a one-to-one correspondence between the set of the idempotent elements of $M(A) = B(G)$ and the coset ring $\mathfrak{R}(G)$ generated by the cosets of the open subgroups of G. It happens that there also exists an abstract version of this theorem of Host [6] so that it is possible to characterize, for an abstract Banach algebra A as above, the invariantly complemented ideals of A in term of certain subsets of the Gelfand spectrum $\Delta(A)$ of A. For this result we refer the reader to the paper [16, Lemma 3.9 and Theorem 3.10].

References

[1] P. Aiena, B.K. Laursen, Multipliers with closed range on regular commutative Banach algebras, Proc. Amer. Math. Soc. 121 (1994) 1039–1408.
[2] G.R. Dales, Banach Algebras and Automatic Continuity, Clarendon Press, Oxford, 2000.
[3] A. Derighetti, Some results on the Fourier–Stieltjes algebra of a locally compact group, Comm. Math. Helv. 45 (1970) 219–228.
[4] P. Eymard, L'algèbre de Fourier d'un groupe locallement compact, Bull. Soc. Math. France 92 (1964) 181–236.
[5] I. Glicksberg, When is $\mu * L^1(G)$ closed? Trans. Amer. Math. Soc. 72 (1971) 419–425.
[6] B. Host, Le théorème des idempotents dans $B(G)$, Bull. Soc. Math. France 114 (1986) 215–223.
[7] B. Host, F. Parreau, Sur un problème de I. Glicksberg: Les idéaux férmés de type fini de $M(G)$, Ann. Inst. Fourier (Grenoble) 28 (1978) 143–164.
[8] C. Herz, Harmonic synthesis for subgroups, Ann. Inst. Fourier (Grenoble) 23 (1973) 91–123.
[9] E. Hewitt, K.A. Ross, Abstract Harmonic Analysis II, Springer, Berlin, 1970.
[10] R. Larsen, The Theory of Multipliers, Springer, Berlin, 1971.

[11] K.B. Laursen, M. Mbekhta, Closed range multipliers and generalized inverses, Studia Math. 107 (1993) 127–135.

[12] K.B. Laursen, M. Neumann, An Introduction to the Local Spectral Theory, Clarendon Press, Oxford, 2000.

[13] A.T. Lau, A. Ülger, Topological centers of certain dual algebras, Trans. Amer. Math. Soc. 384 (1996) 1191–1212.

[14] L. Mate, Embedding multiplier operators of a Banach algebra B into its second conjugate space B^{**}, Bull. Acad. Polan. Sci. Ser. Math. Astronom. Phys. 13 (1965) 809–812.

[15] J.-P. Pier, Amenable Locally Compact Groups, Wiley-Interscience Publication, Florida, 1984.

[16] A. Ülger, Multipliers with closed range on commutative semisimple Banach algebras, Studia Math. 153 (2002) 59–80.

[17] Y. Zaime, Opérateurs de convolution d'image férmé et unité approchées, Bull. Soc. Math. France 99 (1975) 65–74.